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Finite-Time Team-Optimal Strategy with Stochastic H_2/H_∞ Control

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ABSTRACT

This paper discusses the finite-time team-optimal solution for the linear stochastic systems with external disturbances that are attenuated by the H_2/H_∞ control for the problem of multiple decision-makers. The results show that a finite-time team-optimal solution is found by solving a system of coupling stochastic differential Riccati equations (CSDREs). Finally, a real-world example is given to show that the results are correct.

Keywords: Team-optimal solution; H_2/H_∞ control; Linear stochastic system; Differential Riccati equations; State feedback.

1. Introduction

In recent years, the stochastic systems of H_2/H_∞ control problems have concerned much interest. It has been used in many fields, e.g., [1] and its references. For systems operated as the Itô stochastic differential equation, [2] studies the linear stochastic problems with state-dependent noise, [3] performs robust control on Markov jump linear systems, while [4] deals with the discrete-time stochastic system. In this article, we present a stochastic H_2/H_∞ control with a finite-horizon and continuous-time variant state-dependent noise for the Itôs differential system.

Mixed H_2/H_∞ control eliminates the effects of disturbances and is widely used to solve the problem of robust performance control with uncertainty [5, 6]. In engineering applications, to eliminate the effects of interference, while applying the worst-disturbance, it is desirable to minimize the control performance required for control. At a certain disturbance reduction level, the stochastic norm, the perturbation operator norm, and the worst-disturbance have control to the output. Such mixed control problems can be solved by using the Nash game approach [7]. We put forward the idea in the team-optimal control problem for multiple decision-makers, which is more challenging.

When two or more decision-makers collaborate, one of the special categories of non-zero-sum games is the team optimization problem, where the decision-maker has a typical goal. In the general structure, assume that C_r is the cost functional of the M decision-makers. $r_i \in \Omega_i$ is the strategy of the ith decision-maker, $i = 1, 2, \ldots, M$. Then, the team-optimization involves optimizing the cost functional C_r for all $r_i \in \Omega_i$. The subsequent arrangement of the policy set $(r_1^*, r_2^*, \ldots, r_M^*)$ is known as the team-optimal strategy. Our goal is to locate the team-optimal strategy for a class of Itô differential equations. The state-dependent noise is associated with external disturbances, which are attenuated by the H_2/H_∞ control for the finite-time closed-loop feedback system. A group of coupling stochastic differential Riccati equations (CSDREs) is derived to find the required team-optimal solution.

So far, there have been few studies on the issue of team-optimal mixed H_2/H_∞ control. In [8], the optimal Stackelberg strategy for hierarchical control problems is studied. However, it is only regarded as a deterministic system with no external disturbance. Recently, in some research work of incentive Stackelberg games with external disturbance, the idea of team-optimal control problem was adopted, see [9, 10]. However, all studies only consider the H_∞ constraint to reduce the disturbances. Very recent, an infinite-horizon team-optima problem is solved under the H_2/H_∞ control in [11]. The current paper first considers a similar idea for team-optimal control problems in a finite-time stochastic system.

This article will discuss some basic definitions and outcomes based on the H_2/H_∞ control problem. A team-optimal problem for Itô differential equations with several decision-makers is proposed. The results of a set of CSDREs are derived in detail for the continuous finite-time case. A realistic numerical model illustrates our proposed method.

The symbols in this article are standard, as follows:

X^T: transpose of matrix or vector X;

$X > 0$ $(X < 0)$: positive (negative) definite symmetric matrix X;

$X \geq 0$ $(X \leq 0)$: positive (negative) semidefinite symmetric matrix X;

$\|\cdot\|$: the Euclidean norm;

I_n: an $n \times n$ identity matrix;

$\mathbb{E}[\cdot]$: the expectation operator;

$\mathcal{L}_F^2([0, T], \mathbb{R}^\ell)$: the space of random process $u(t) \in \mathbb{R}^l$ regarding a rising σ-algebras $F_t(t \geq 0)$ with $\mathbb{E}\left[\int_0^T \| u(t) \|^2 \, dt\right] < \infty$, where T denotes the final value of time. For convenience, the time argument (t) is often omitted.

2. Preliminaries

Here, we will discuss some primary results on H_2/H_∞ control problem. First, we discuss the concept of the team-optimal strategy [8].

***Definition 1*:** Let $C_0(r_1, r_2, \dots, r_M)$ be a typical cost functional of M players, where r_i, $i = 1, 2, \dots, M$ denotes the ith player's control. If

$$C_0(r_1^*, r_2^*, \dots, r_M^*) \le C_0(r_1, r_2, \dots, r_M), \quad (1)$$

for any r_i, $i = 1, 2, \dots, M$, the control strategy $(r_1^*, r_2^*, \dots, r_M^*)$ is said to be a team-optimal strategy.

Secondly, we will introduce a finite-time stochastic H_2/H_∞ control problem as follows:

$$\begin{aligned} dx(t) &= [X(t)x(t) + Y(t)r(t) \\ &\quad + G(t)\tau(t)]dt + X_s(t)x(t)dw(t), \\ x(0) &= x_0, \\ z(t) &= [D(t)x(t) \quad E(t)r(t)]^T, \\ E^T(t)E(t) &= I_m, \end{aligned} \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $r(t) \in \mathcal{L}_F^2([0, T], \mathbb{R}^m)$, $\tau(t) \in \mathcal{L}_F^2([0, T], \mathbb{R}^{n_\tau})$, $w(t) \in \mathbb{R}$, $z(t) \in \mathbb{R}^{n_z}$ are denoted the state vector, control input, external deterministic disturbance, one-dimensional Brownian motion, and controlled output respectively. The coefficients $X(t)$, $Y(t)$, $D(t)$, $E(t)$, $G(t)$, $X_s(t)$ depend on time having appropriate dimensions. The finite-time H_2/H_∞ control problem can be defined as below [2].

***Definition 2*:** Considering the external disturbance reduction stage $\gamma > 0$, the H_2/H_∞ control problem requires to get a state feedback control $r^*(t) = K(t)x(t)$ and a worst-disturbance $\tau^*(t) = F(t)x(t)$ so that

• If $r^*(t) = K(t)x(t)$ is applied in the equation (2), the perturbation operator $\| \mathcal{L} \|_{[0,T]}$ of (2) can be determined as follows:

$$\| \mathcal{L} \|_{[0,T]} = \sup_{\substack{\tau \in \mathcal{L}_F^2([0,T],\mathbb{R}^{n_\tau}) \\ \tau \neq 0, x^0 = 0}} \frac{\| z \|_{[0,T]}}{\| \tau \|_{[0,T]}}$$

$$= \sup_{\substack{\tau \in \mathcal{L}_F^2([0,T],\mathbb{R}^{n_\tau}) \\ \tau \neq 0, x^0 = 0}} \frac{\sqrt{C_r(r^*)}}{\sqrt{\mathbb{E}\left[\int_0^T \|\tau(t)\|^2 dt\right]}} < \gamma, \quad (3)$$

where

$$C_r(r) := \mathbb{E}\left[\int_0^T \| z(t) \|^2 \, dt\right]$$

$$= \mathbb{E}\left[\int_0^T \begin{Bmatrix} x^T(t)D^T(t)D(t)x(t) \\ + r^T(t)r(t) \end{Bmatrix} dt\right]. \quad (4)$$

• If $\tau^*(t) = F(t)x(t)$ is employed in the system (2), and $r^*(t)$ minimizes the cost, then $C_r(r, \tau^*) = C_r(r)$ in (4). The worst-disturbance

$$\tau^*(t) = \arg min_\tau \, C_\tau(r^*, \tau), \quad (5)$$

where

$$C_\tau(r^*, \tau) = \mathbb{E}\left[\int_0^T \{\gamma^2 \| \tau(t) \|^2 - \| z(t) \|^2\} dt\right]. \quad (6)$$

***Lemma 1*:** [2] Consider the stochastic system

$$\begin{aligned} dx(t) &= \big(X(t)x(t) + G(t)\tau(t)\big)dt \\ &\quad + X_s(t)x(t)dw, \\ z(t) &= D(t)x(t), \end{aligned} \quad (7)$$

where $x(0) = x_0 \in R^n, z \in \mathcal{L}_F^2(R_+, R^{n_z})$.

The perturbation operator is define as $\mathcal{L}(\tau) = D(t)x(t), t \geq 0, \tau \in \mathcal{L}_F^2(R_+, R^{n_\tau})$

$$\| \mathcal{L} \|_\infty = \sup_{\substack{\tau \in \mathcal{L}_F^2([0,T],\mathbb{R}^{n_\tau}) \\ \tau \neq 0, x_0 = 0}} \frac{\| z \|_2}{\| \tau \|_2}$$

$$= \sup_{\substack{\tau \in \mathcal{L}_F^2([0,T],\mathbb{R}^{n_\tau}) \\ \tau \neq 0, x_0 = 0}} \frac{\left\{\mathbb{E}\int_0^\infty x^T D^T D x dt\right\}^{1/2}}{\left\{\mathbb{E}\int_0^\infty \tau^T \tau dt\right\}^{1/2}} \quad (8)$$

is $\| \mathcal{L} \|_{[0,T]} < \gamma$ for some $\gamma > 0$ if and only if the following SDRE

$$\begin{aligned} &\dot{V} + VX + X^T V + X_s^T V X_s - \gamma^{-2} V G G^T V \\ &- D^T D = 0, \quad V(T) = 0, \end{aligned} \quad (9)$$

has a solution $V(t) \le 0$ on $[0, T]$.

3. Problem Formulation

let consider the subsequent Itô stochastic differential equation with multiple controls:

$$\begin{aligned} dx(t) &= \left[X(t)x(t) + \sum_{i=1}^M Y_i(t)r_i(t) + G(t)\tau(t)\right]dt \\ &\quad + X_s(t)x(t)dw(t), x(0) = x_0, \\ z(t) &= [D(t)x(t) \quad r_1(t) \quad r_2(t) \quad \cdots \quad r_M(t)]^T, \end{aligned} \quad (10)$$

where, $r_i(t) \in \mathcal{L}_F^2([0, T], \mathbb{R}^{m_i})$, $i = 1, 2, , M$ represents the control input of i^{th} decision-maker. The coefficient matrices are time-variant with suitable dimensions, and other notations represent the same meaning as the previous section. The finite-time team-optimal stochastic H_2/H_∞ control problem of the system (10) can be declared as follows.

For the given disturbance reduction stage $\gamma > 0, 0 \le T$ we have to locate the state feedback control $r_i^*(t) = K_i(t)x(t)$, $i = 1, 2, \dots, M$ and worst-disturbance $\tau^*(t) = F(t)x(t)$ so that

i) If $r_i^*(t) = K_i(t)x(t)$, $i = 1, 2, \dots, M$ is applied in the system (10), then the perturbation operator $\| \mathcal{L} \|_{[0,T]}$ of (10) can be determined as follows:

$$\| \mathcal{L} \|_{[0,T]} = \sup_{\substack{\tau \in \mathcal{L}_F^2([0,T],\mathbb{R}^{n_\tau}) \\ \tau \neq 0, x^0 = 0}} \frac{\| z \|_{[0,T]}}{\| \tau \|_{[0,T]}}$$

$$= \sup_{\substack{\tau \in \mathcal{L}_F^2([0,T],\mathbb{R}^{n_\tau}) \\ \tau \neq 0, x^0 = 0}} \frac{\left\{\mathbb{E}\int_0^T \begin{Bmatrix} x^T(t)D^T D(t)x(t) \\ + \sum_{i=1}^M [r_i^T(t)r_i(t)] \end{Bmatrix} dt\right\}^{\frac{1}{2}}}{\left\{\mathbb{E}\left[\int_0^T \| \tau(t) \|^2 \, dt\right]\right\}^{\frac{1}{2}}}$$

$$< \gamma. \quad (11)$$

ii) While the worst-disturbance $\tau^*(t) = F(t)x(t)$ (if it exists) is employed to (10), $r_i(t), i = 1, 2, \dots, M$ minimizes the team cost

$$C_r(r_1, r_2, \dots, r_M, \tau^*) = \mathbb{E}\int_0^T (x^T D^T D x + \sum_{i=1}^M r_i^T r_i) dt. \quad (12)$$

Here,

$$\tau^*(t) = arg\ min_\tau\ C_\tau(r_1^*, r_2^*, \dots, r_M^*, \tau), \quad (13)$$

Where

$$C_\tau(r_1^*, r_2^*, \dots, r_M^*, \tau) = \mathbb{E}\int_0^T (\gamma^2 \tau^T \tau - z^T z) dt, \quad (14)$$

$\forall \tau(t) \in \mathcal{L}_F^2([0,T], \mathbb{R}^{n_\tau})$. If such $(r_1^*, r_2^*, \dots, r_M^*, \tau^*)$ exists, we call it the finite-time team-optimal H_2/H_∞ control.

Now we state the necessary and sufficient conditions for the solution set $(r_1^*, r_2^*, \dots, r_M^*, \tau^*)$, which is equivalent to find the Nash equilibrium conditions:

$$\begin{cases} C_r^T(r_1^*, r_2^*, \dots, r_M^*, \tau^*) \le C_r^T(r_1, r_2, \dots, r_M, \tau^*), \\ C_\tau^T(r_1^*, r_2^*, \dots, r_M^*, \tau^*) \le C_\tau^T(r_1^*, r_2^*, \dots, r_M^*, \tau). \end{cases} \quad (15)$$

4. Main Results

We rearrange the system (10) to the following form:

$$dx(t) = \left[X(t)x(t) + Y_f(t)r_f(t) + G(t)\tau(t)\right]dt$$

$$+X_s(t)x(t)dw(t), x(0) = x_0, \quad (16)$$

$$z(t) = [D(t)x(t) \quad r_f(t)]^T,$$

where,

$$Y_f = [Y_1, Y_2, \dots, Y_M],\ \ r_f = [r_1, r_2, \dots, r_M]^T.$$

Here, $r_f(t) \in \mathcal{L}_F^2([0,T], \mathbb{R}^{\sum_{i=1}^M m_i})$ represents the control input of a team of M decision-makers. The following theorem can be obtained by making necessary modifications to Theorem 5 of [2].

***Theorem 1*:** For the system (16) the finite -time H_2/H_∞ team-optimal feedback strategies can be found as follows:

$$\begin{cases} r_f^*(t) &= K(t)x(t), \\ \tau^*(t) &= F(t)x(t), \end{cases} \quad (17)$$

iff the following CSDREs:

$$-\dot{U} = X^T U + UX + X_s^T U X_s - D^T D - SU$$

$$-\left(UY_fY_f^T + VY_fY_f^T\right)V,\ \ U(T) = 0 \quad (18a)$$

$$-\dot{V} = X^T V + VX + X_s^T V X_s + D^T D - V\gamma^{-2}GG^T U$$

$$-SV,\ \ V(T) = 0, \quad (18b)$$

with

$$S = U\gamma^{-2}GG^T + VY_fY_f^T,$$

have solutions $U(t) \le 0, V(t) \ge 0$ on $[0,T]$. In this case,

(i) $r_f^*(t) = K(t)x(t) = -Y_f^T V(t)x(t)$, $\tau^*(t) = F(t) = -\gamma^{-2}G^T U(t)x(t)$, where

$$\begin{cases} K(t) &= -Y_f^T V(t), \\ F(t) &= -\gamma^{-2}G^T U(t). \end{cases} \quad (19)$$

(ii) $C_r(r_f^*, \tau^*) = x_0^T V x_0$.

***Proof*:** *Sufficient condition*: If we apply Itô's formula with the constraint of (16), we have

$$C_\tau(r,\tau) = x_0^T U(0)x_0 - \mathbb{E}(x^T(T)U(T)x(T)) + \mathbb{E}\int_0^T [(\gamma^2 r_f^T r_f - z^T z)dt + d(x^T U(t)x)]$$

$$= x_0^T U(0)x_0 + \mathbb{E}\int_0^T [(\gamma^2 r_f^T r_f - z^T z)dt$$

$$+x^T \dot{U}(t)x + (dx)^T U(t)x + x^T U(t)x$$

$$+x^T U(t)dx(dx)^T U(t)dx] \quad (20)$$

By computing with (18a), the equation (20) can be written as

$$C_\tau(r,\tau) - x_0^T U(0)$$

$$= \mathbb{E}\int_0^T [\gamma^2(\tau-\tau^*)^T(\tau-\tau^*) - r_f^T r_f + r_T^{*T} r^*$$

$$+2x^T U(t)Y_f(r_f(t) - r_f^*(t,x))]dt. \quad (21)$$

Implementi$r_f(t) = r_f^*(t,x) = -Y_f^T V(t)x(t)$ in (21) follows:

$$C_\tau(r_f^*, \tau) \ge C_\tau(r_f^*, \tau^*) = x_0^T U(0)x_0, \quad (22)$$

where $\tau^*(t,x) = -\gamma^{-2}GU(t)x(t)$. When $x_0 = 0$, then

$$C_\tau(r_f^*, \tau) = \mathbb{E}\int_0^T \gamma^2(\tau-\tau^*)^T(\tau-\tau^*)dt. \quad (23)$$

According to the definition of $\tau^*(t)$ in (13), by completing the square and considering (18b), we get

$$C_{r_f}(r_f, \tau^*) = C_{r_f}(r_f^*, \tau^*) = x_0^T V x_0. \quad (24)$$

According to Definition 2, the finite-time H_2/H_∞ control has a strategy set (r_f^*, τ^*) with

$$\begin{cases} r_f^*(t) &= K(t)x(t) = -Y_f^T V(t)x(t), \\ \tau^*(t) &= F(t)x(t) = -\gamma^{-2}G^T U(t)x(t). \end{cases} \quad (25)$$

Necessary condition: Implement $r_f^*(t) = K(t)x(t)$ in (10), then

$$dx(t) = \Big(X(t) + Y_f(t)K(t)x(t) + G(t)\tau(t)\Big)dt$$

$$+X_s(t)x(t)dw, \quad x(0) = x_0, \quad (26)$$

$$z(t) = \begin{bmatrix} D(t)x(t) \\ E(t)K(t)x(t) \end{bmatrix}, E^T(t)E(t) = I.$$

By Definition 2 $\| \mathcal{L} \|_{[0,T]} < \gamma$. By Lemma 1, the following SDRE:

$$\dot{U} + U(X + Y_fK) + (X + Y_fK)^T U + X_s^T U X_s$$

$$-\gamma^{-2}UGG^T U - D^T D - K^T K = 0, \quad (27)$$

$U(T) = 0$, has a solution $U(t) \leq 0$ on $[0, T]$. By Lemma 1, we can also see that $\tau^*(t) = F(t)x(t) = -\gamma^{-2}G^T U(t)x(t)$, where $U(t)$ solves (27). Substituting $\tau = \tau^* = -\gamma^{-2}G^T U(t)x(t)$ into (16) yields

$$dx(t) = \big(X(t) - \gamma^{-2}G(t)G^T(t)U(t)\big)x(t)$$

$$+(Y_f(t)r_f(t))dt + X_s(t)xdw, x(0) = x_0, \quad (28)$$

$$z(t) = \begin{bmatrix} D(t)x(t) \\ E(t)r_f(t) \end{bmatrix}, E^T(t)E(t) = I,$$

How to minimize $C_r(r_f, \tau^*)$ along with (27) is standard optimal regulator problem. The following SDRE:

$$\dot{V} + V(X - \gamma^{-2}GG^T U) + (X - GG^T U)^T V + X_s^T V X_s$$

$$-VY_f Y_f^T V + D^T D = 0, \quad (29)$$

$V(T) = 0$, has a positive-semidefinite solution $V(t) \geq 0$ on $[0, T]$. Substituting $V(t)$ into (29) follows (18b). Furthermore, by the computation perfect square, we have

$$C_r(r_f, \tau^*) = x_0^T V(0)x_0 + \mathbb{E}\int_0^T [(r_f + Y_f^T Vx)^T$$

$$\times (r_f + Y_f^T Vx)]dt. \quad (30)$$

Therefr$_f^*(t, x) = K(t)x(t) = -Y_f^T V(t)x(t)$ solves the following stochastic H_2 optimization problem:

$$\min_{r_f \in F_F^2(R_+, R^n)} C_{r_f}(r_f, \tau^*) = C_{r_f}(r_f^*, \tau^*)$$

$$= x_0^T V x_0. \quad (31)$$

Substituting $U(t) \leq 0$ and $K = -Y_f^T V(t)$ into (27), (18) is obtained. Hence, the theorem is proved.

To compute the team-optimal solution, the flowchart presented in Fig. 1 can be followed.

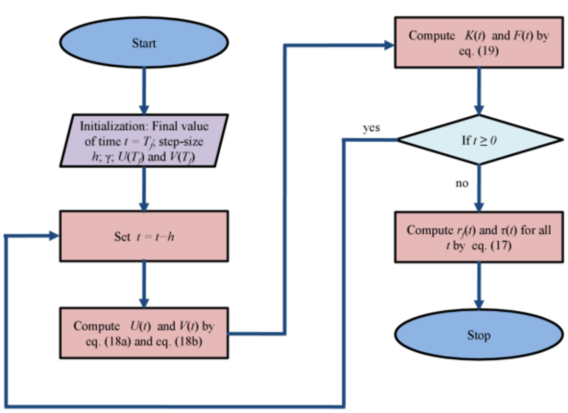

Fig. 1: Flowchart for finding team-optimal solution followed by Theorem 1

5. Numerical Example

Consider an improved knowledge investment inventory form the reference [12] with a slide modification. The details are that two people are investing in the public domain of knowledge. Due to the influence of noise and external interference, the knowledge stock evolves regarding the stochastic differential equation (10) with $x_0 = [0.5 \ -0.25]^T$. Here $x(t)$, and $r_i(t)$, where $i = 1,2$ are denoted the stock of knowledge, and the investment of ith player in the public knowledge at t, correspondingly. The coefficients of the system are shown below:

$$X(t) = \begin{bmatrix} -t - 0.5 & 0.5\text{cos}t \\ 0 & -1.5 \end{bmatrix},$$

$$Y_1(t) = \begin{bmatrix} -0.3 \\ -0.5 \end{bmatrix}, Y_2(t) = \begin{bmatrix} -\text{sin}t \\ -1 \end{bmatrix},$$

$$G(t) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, X_s(t) = \begin{bmatrix} \text{sin}t & 0.5 \\ 0 & \text{tanh}t \end{bmatrix},$$

$$D(t) = [0.2 \quad -0.8].$$

Set the final value of time $T = 20$, time step-size $h = 0.1$, disturbance reduction level $\gamma = 0.8$, $U(T) = 0$ and $V(T) = 0$ in the CSDREs (18) of Theorem 1. The CSDREs (18) are solved for $U(t)$ and $V(t)$ by using the Runge-Kutta method in Matlab. The symmetric 2×2 matrices $U(t)$ and $V(t)$ are exposed in Fig. 2.

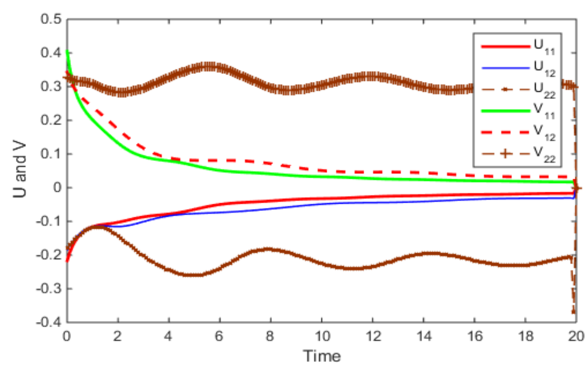

Fig. 2: Solutions $U(t)$ and $V(t)$ of the equations (18a) and (18b)

To find the team-optimal solution the evaluated gain matrices $K(t)$ and $F(t)$ of (19) are exposed in Fig. 3.

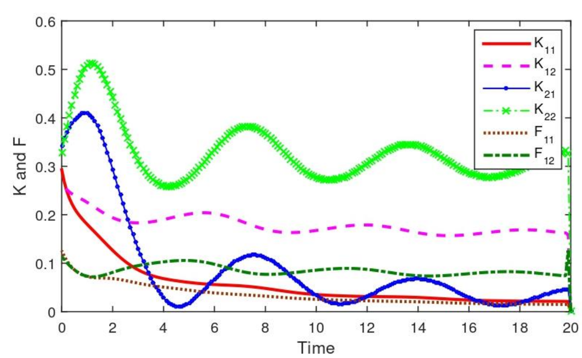

Fig. 3: Evaluation of $K(t)$ and $F(t)$ by equation (19)

Finally, we can draw the optimal state trajectory shown in Fig. 4. The figure is generated by applying the finite difference method over (10). It can be observed that both components of the state vector $x(t)$ converge to zero at near about time $t = 4$. Fig. 4 represents that the state variable $x(t)$ is stable for the system (10), which means that the current technique is consistent to use.

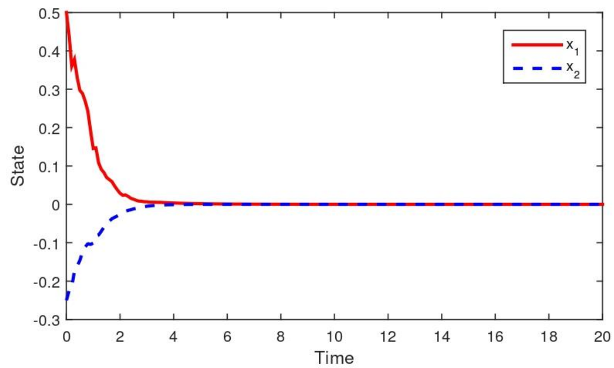

Fig. 4: Optimal state trajectory tested by equation (10)

6. Conclusion

The team-optimal solution has been investigated under H_2/H_∞ control for the stochastic systems with more than one controls with the Itô differential equation. Unlike the existing deterministic system, the stochastic system involving white noise and exogenous disturbance is studied for the first time in the current study. It should be noted that the strategy set for the solution of this article can be found by solving a group of CSDREs. Finally, a real-life numerical model has been explored to demonstrate the viability and usefulness of finding the team-optimal solution with mixed H_2/H_∞ control.

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References

1. W. Zhang, H. Zhang and B.S. Chen, "Stochastic H_2/H_∞ control with (x, u, v)-dependent noise: finite-horizon case", *Automatica*, vol. 42, no. 11, pp 1891–1898, 2006.
2. B.S. Chen and W. Zhang, "Stochastic H_2/H_∞ control with state-dependent noise", *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp 45–57, 2004.
3. Y. Huang, Y. Zhangx and G. Feng, "Infinite horizon H_2/H_∞ control for stochastic systems with Markovian jumps", *Automatica*, vol. 44, no. 3, pp 857–863, 2008.
4. W. Zhang, Y. Huang and H. Zhang, "Stochastic H_2/H_∞ control for discrete-time systems with state and disturbance dependent noise", *Automatica*, vol. 43, no. 3, pp 513–521, 2007.
5. C.C. Ku and C.I. Wu, "Gain-scheduled H_∞ control for linear parameter varying stochastic systems", *Journal of Dynamic Systems Measurement and Control*, vol. 137, no. 11, pp 111012-1–12, 2015.
6. H. Mukaidani, "Gain-scheduled H_∞ constraint Pareo optimal strategy for stochastic LPV systems with multiple decision makers", *American Control Conference (ACC)*, Seattle, WA, USA, pp 1097–1102, 2017.
7. H. Mukaidani, "Finite-Horizon Closed-Loop Nash Game for Stochastic Large-Scale Systems with Multiple Decision Makers", *Proc. American Control Conf.*, Chicago, IL, pp 1481–1486, 2015.
8. T. Başar, G.J. Olsder, "Team-optimal closed-loop Stackelberg strategies in hierarchical control problems", *Automatica*, no. 16, vol. 4, pp 409–414, 1980.
9. M. Ahmed, H. Mukaidani and T. Shima, "H_∞-constrained incentive Stackelberg games for discrete-time stochastic systems with multiple followers", *IET Control Theory & Applications*, vol. 11, no. 15, pp 2475–2485, 2017.
10. M. Ahmed, H. Mukaidani and T. Shima, "Infinite-horizon multi-leader-follower incentive Stackelberg games for linear stochastic systems with H_∞ constraint", *56th Annual Conference of the Society of Instrument and Control Engineers (SICE)*, Kanazawa, Japan, pp 1202–1207, 2017.
11. M. Ahmed, M.A.B. Masud and M.R. Karim, "Team-optima stochastic H_2/H_∞ control with multiple decision makers", *Jagannath University Journal of Science*, vol. 6, no. I & II, pp 81–91, 2020.
12. J.C. Engwerda, "*LQ Dynamic Optimization and Differential Games*", Chichester: Wiley, 2005.